

## Perfect Fluidity in Atomic Physics

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**Abstract.** Experimental results obtained at the Relativistic Heavy Ion Collider (RHIC) have been interpreted in terms of a strongly interacting quark gluon plasma. The strongly interacting plasma is characterized by “perfect fluidity”, i.e. a ratio of shear viscosity to entropy density that saturates a proposed lower bound. In this contribution we explore the possibility that a similar phenomenon takes place in a strongly coupled non-relativistic Fermi liquid in which the scattering length between the Fermions is infinitely large.

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### 1. Introduction

Experiments at the Relativistic Heavy Ion Collider (RHIC) indicate that a new state of matter is produced in high energy heavy ion collisions [ 1]. Much effort is currently devoted to characterizing the properties of this state, and to determining the nature of its low energy excitations.

Asymptotic freedom implies that the equation of state of a quark gluon plasma at  $T \gg \Lambda_{QCD}$  is that of a free gas of quarks and gluons. Numerical results from lattice QCD calculations show that at  $T \sim 2T_c$ , which is relevant to the early stages of heavy ion collisions at RHIC, the pressure and energy density reach about 85% of the free gas limit. The 15% reduction is consistent with the magnitude of the first order perturbative correction. Higher order terms in the perturbative expansion converge very slowly, but this problem can be overcome using resummation techniques [ 2]. In this framework the degrees of freedom are dressed quasi-quarks and quasi-gluons, and these quasi-particles are weakly interacting.

Transport properties of the matter created at RHIC indicate that this may not be correct. Experiments at RHIC suggest that the viscosity of the plasma is very small, and that the opacity for high energy jets is very large. If the plasma is composed of weakly interacting quasi-particles then the shear viscosity can be

estimated using kinetic theory. The result is

$$\eta \simeq \frac{1}{3} n p l, \quad (1)$$

where  $n$  is the density,  $p$  is the average momentum, and  $l$  is the mean free path. In a relativistic system the number of particles is not conserved, and it is more natural to express the viscosity in units of the entropy density rather than the density. Since the entropy per particle (in units of  $k_B$ ) is of order one, this does not qualitatively change the numerical coefficient in equ. (1). A good quasi-particle is characterized by a small ratio of the width over the excitation energy. This implies that the product of the mean free path times the typical momentum is large, and  $\eta/s$  is big. This is confirmed by a weak coupling calculation in perturbative QCD. Arnold et al. find [ 3]

$$\frac{\eta}{s} = \frac{5.12}{g^4 \log(2.42g^{-1})}, \quad (2)$$

where  $g$  is the strong coupling constant. For  $\alpha_s \leq 1/3$  we get  $\eta/s > 1.75$ . This result should be contrasted with the values extracted at RHIC, which are in the range  $\eta/s < 0.5$  [ 4, 5].

From a theoretical point of view the RHIC results raise the question of how small the viscosity can get. Clearly,  $\eta/s$  decreases as the interaction becomes stronger but there are good reasons to believe that the shear viscosity always remains finite. In particular, it seems reasonable to assume that the product of the mean free path and the typical momentum cannot be smaller than  $\hbar$  [ 6]. An interesting perspective on this issue is provided by a strong coupling calculation performed in the large  $N_c$  limit of  $\mathcal{N} = 4$  SUSY Yang Mills theory. The calculation is based on the duality between strongly coupled gauge theory and weakly coupled string theory on  $AdS_5 \times S_5$  discovered by Maldacena [ 7]. The thermodynamics of the SUSY field theory was studied by Gubser et al. [ 8]. They find that the entropy density in the strong coupling limit is 3/4 of the free field theory result. This implies that the equation of state is not drastically affected by the value of the coupling. The calculation was extended to transport properties by Policastro et al. [ 9]. These authors find that the shear viscosity to entropy density ratio of the strongly coupled gauge theory is  $\eta/s = \hbar/(4\pi)$ . This number is quite consistent with the values extracted from RHIC data. Kovtun et al. studied the behavior of  $\eta/s$  in other strongly coupled field theories with gravity duals and conjectured that the value  $\hbar/(4\pi)$  is a universal lower bound for  $\eta/s$  [ 10, 11].

## 2. Cold atomic gases

In order to understand the relevance of these results to the RHIC data it is useful to study the transport properties of other strongly coupled fluids that are experimentally accessible. Over the last ten years there has been truly remarkable progress in the study of cold, dilute gases of fermionic atoms in which the scattering length

$a$  of the atoms can be controlled experimentally. These systems can be realized in the laboratory using Feshbach resonances, see [ 12] for a review. A small negative scattering length corresponds to a weak attractive interaction between the atoms. This case is known as the BCS (Bardeen-Cooper-Schrieffer) limit. As the strength of the interaction increases the scattering length becomes larger. The scattering length diverges at the point where a bound state is formed. This is called the unitarity limit, because the scattering cross section saturates the  $s$ -wave unitarity bound  $\sigma = 4\pi/k^2$ . On the other side of the resonance the scattering length is positive. In the BEC (Bose-Einstein condensation) limit the interaction is strongly attractive and the fermions form deeply bound molecules.

The unitarity limit is of particular interest. In this limit the atoms form a strongly coupled quantum liquid which exhibits universal behavior. At zero temperature the atomic liquid is characterized by the mass of the atoms  $m$ , the density  $n$ , the scattering length  $a$ , and the effective range  $r$ . A dilute gas at unitarity corresponds to the limit in which  $a^3n \rightarrow \infty$  and  $r^3n \rightarrow 0$ . This implies that the dependence of a physical observable on  $n$  and  $m$  is determined by simple dimensional analysis, but the overall numerical constant is a complicated, non-perturbative quantity. At finite temperature  $T$  properties of the fluid are universal functions of the dimensionless variable  $T/T_F$ , where  $T_F \sim n^{2/3}/m$ .

In cold atomic gases we can reliably compute  $\eta/s$  in the BCS limit [ 13]. The ratio is temperature dependent and has a minimum at  $T \sim T_F$ . The shear viscosity is proportional to  $1/a^2$ , and  $\eta/s$  is very large in the weak coupling limit. As in the case of QCD there are no controlled calculations in the strong coupling limit  $a \rightarrow \infty$ . It is possible, however, to reliably extract  $\eta/s$  from experimental data on the damping of collective oscillations [ 14, 15, 16].

### 3. Collective Oscillations

In the strong coupling limit we can assume that collective modes are approximately described by ideal fluid dynamics. The equation of continuity and of momentum conservation are given by

$$\frac{\partial n}{\partial t} + \vec{\nabla} \cdot (n\vec{v}) = 0, \quad (3)$$

$$mn \frac{\partial \vec{v}}{\partial t} + mn (\vec{v} \cdot \vec{\nabla}) \vec{v} = -\vec{\nabla} P - n\vec{\nabla} V, \quad (4)$$

where  $n$  is the number density,  $m$  is the mass of the atoms,  $\vec{v}$  is the fluid velocity,  $P$  is the pressure and  $V$  is the external potential. The trapping potential is harmonic

$$V = \frac{m}{2} \sum_i \omega_i^2 r_i^2. \quad (5)$$

Universality implies that the equation of state is given by  $P = n^{\gamma+1} f(T/T_F)$  with  $\gamma = 2/3$ . The compressibility at constant entropy is

$$\left( \frac{\partial P}{\partial n} \right)_S = (\gamma + 1) \frac{P}{n}. \quad (6)$$

The equilibrium distribution  $n_0$  can be determined from the hydrostatic equation  $\vec{\nabla} P_0 = -n_0 \vec{\nabla} V$ . At  $T = 0$

$$n_0(\vec{r}) = n_0(0) \left( 1 - \sum_i \frac{r_i^2}{R_i^2} \right)^{1/\gamma}, \quad R_i^2 = \frac{2\mu}{m\omega_i^2}, \quad (7)$$

where  $\mu$  is the chemical potential. In the unitarity limit the chemical potential is related to the Fermi energy as  $\mu = \xi E_F$ , where  $\xi$  is a universal parameter. Green function Monte Carlo calculations give  $\xi \simeq 0.44$  [17]. We consider small oscillations  $n = n_0 + \delta n$ . From the linearized continuity and Euler equation we get [18]

$$m \frac{\partial^2 \vec{v}}{\partial t^2} = -\gamma (\vec{\nabla} \cdot \vec{v}) (\vec{\nabla} V) - \vec{\nabla} (\vec{v} \cdot \vec{\nabla} V), \quad (8)$$

where we have dropped terms of the form  $\nabla_i \nabla_j \vec{v}$  that involve higher derivatives of the velocity. This equation has simple scaling solutions of the form  $v_i = a_i x_i \exp(i\omega t)$  (no sum over  $i$ ). Inserting this ansatz into equ. (8) we get an equation that determines the eigenfrequencies  $\omega$ . The experiments are performed using a trapping potential with axial symmetry,  $\omega_1 = \omega_2 = \omega_0$ ,  $\omega_3 = \lambda \omega_0$ . In this case we find one solution with  $\omega^2 = 2\omega_0^2$  and two solutions with [18, 19, 20]

$$\omega^2 = \omega_0^2 \left\{ \gamma + 1 + \frac{\gamma + 2}{2} \lambda^2 \pm \sqrt{\frac{(\gamma + 2)^2}{4} \lambda^4 + (\gamma^2 - 3\gamma - 2) \lambda^2 + (\gamma + 1)^2} \right\}. \quad (9)$$

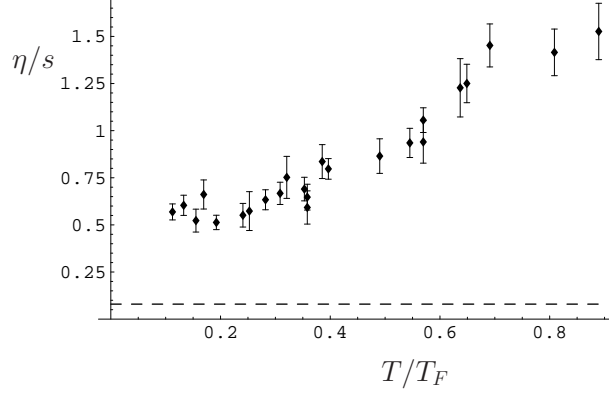
In the unitarity limit ( $\gamma = 2/3$ ) and for a very asymmetric trap,  $\lambda \rightarrow 0$ , the eigenfrequencies are  $\omega^2 = 2\omega_0^2$  and  $\omega^2 = (10/3)\omega_0^2$ . The mode  $\omega^2 = (10/3)\omega_0^2$  is a radial breathing mode with  $\vec{a} = (a, a, 0)$  and the mode  $\omega^2 = 2\omega_0^2$  corresponds to a radial dipole  $\vec{a} = (a, -a, 0)$ .

The frequency of the radial breathing mode agrees very well with experimental results [21]. In a hydrodynamic description the damping of collective modes is due to viscous effects. The dissipated energy is given by

$$\dot{E} = -\frac{1}{2} \int d^3x \eta(x) \left( \partial_i v_j + \partial_j v_i - \frac{2}{3} \delta_{ij} \partial_k v_k \right)^2 - \int d^3x \zeta(x) (\partial_i v_i)^2, \quad (10)$$

where  $\eta$  is the shear viscosity and  $\zeta$  is the bulk viscosity. In the unitarity limit the system is scale invariant and the bulk viscosity in the normal phase vanishes. For the radial scaling solutions we get

$$\overline{E} = -\frac{2}{3} (a_x^2 + a_y^2 - a_x a_y) \int d^3x \eta(x), \quad (11)$$



**Fig. 1.** Viscosity to entropy density ratio of a cold atomic gas in the unitarity limit, from [16]. This plot is based on the damping data published in [23] and the thermodynamic data in [24, 25]. The dashed line shows the conjectured viscosity bound  $\eta/s = 1/(4\pi)$ .

where  $\overline{E}$  is a time average. The damping rate is given by the ratio of the energy dissipated to the total energy of the collective mode. The kinetic energy is

$$E_{kin} = \frac{m}{2} \int d^3x n(x) \overline{v}^2 = \frac{mN}{2} (a_x^2 + a_y^2) \langle x^2 \rangle. \quad (12)$$

At  $T = 0$  we find  $\langle x^2 \rangle = R_\perp^2/8$ , where  $R_\perp$  is the transverse size of the cloud. At non-zero temperature we can use the Virial theorem [22] to relate  $\langle x^2 \rangle$  to the total energy of the equilibrium state,  $\langle x^2 \rangle / \langle x^2 \rangle_{T=0} = E/E_{T=0}$ . The damping rate is [14, 15]

$$-\frac{1}{2} \frac{\overline{E}}{E} = \frac{2}{3} \frac{a_x^2 + a_y^2 - a_x a_y}{a_x^2 + a_y^2} \frac{\int d^3x \eta(x)}{mN \langle x^2 \rangle}. \quad (13)$$

Note that the second factor on the RHS is 1/2 for the radial breathing mode and 3/2 for the radial dipole mode. If this dependence could be demonstrated experimentally, it would confirm that the damping is indeed dominated by shear stress.

We shall assume that the shear viscosity is proportional to the entropy density,  $\eta(x) = \alpha s(x)$ . We note that since the flow profile has a simple scaling form the damping rate is proportional to the volume integral of the shear viscosity. If  $\eta \sim s$  then the damping rate is proportional to the total entropy. The kinetic energy, on the other hand, scales with the number of particles. We can now relate the dimensionless damping rate  $\Gamma/\omega_\perp = 1/(\tau\omega_\perp)$  of the radial breathing mode to the shear viscosity to entropy density ratio. We find

$$\frac{\eta}{s} = \frac{3}{4} \xi^{1/2} (3N)^{1/3} \left( \frac{\overline{\omega}\Gamma}{\omega_\perp^2} \right) \left( \frac{E}{E_{T=0}} \right) \left( \frac{N}{S} \right). \quad (14)$$

Fig. 1 shows  $\eta/s$  extracted from the experimental results of the Duke group [23]. The entropy per particle was also taken from experiment [25]. Similar results are obtained if the entropy is extracted from quantum Monte Carlo data. The critical temperature for the superfluid/normal transition is  $T_c/T_F \simeq 0.29$ . We observe that  $\eta/s$  in this regime is roughly 1/2. This value is compatible with the conjectured viscosity bound and comparable to the values that have been extracted at RHIC.

#### 4. Elliptic Flow

Collective modes are very useful because it is possible to track many compression and expansion cycles and even small damping coefficients can be measured. In heavy ion collisions we do not have this luxury and we have to rely on collective flow measurements to extract transport coefficients. In the following we shall estimate the effect of a non-zero shear viscosity on the elliptic flow of a cold atomic liquid. We consider the expansion of the atomic cloud after the trapping potential is removed. The expansion is described by a simple scaling ansatz

$$n(r_i, t) = n_0(r_i/b_i(t)) \quad (i = 1, \dots, 3), \quad v_i(\vec{r}, t) = \frac{\dot{b}_i(t)r_i}{b_i(t)}. \quad (15)$$

It is easy to check that this ansatz satisfies the continuity equation. The Euler equation gives [26]

$$\ddot{b}_i = \frac{\omega_i^2}{(b_1 b_2 b_3)^\gamma} \frac{1}{b_i}. \quad (16)$$

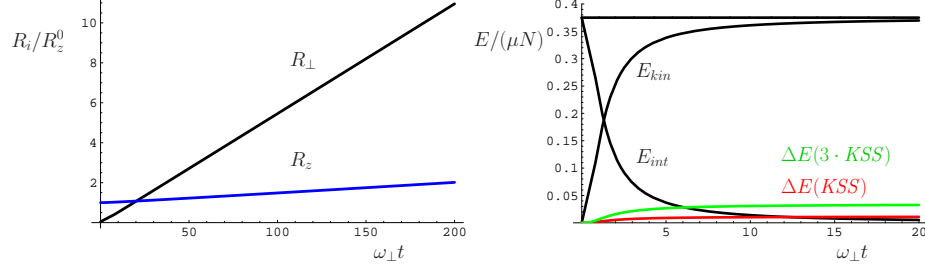
We are interested in an axially symmetric trap with  $\omega_x = \omega_y = \omega_\perp$  and  $\omega_z = \lambda \omega_\perp$ . In the limit  $b_\perp \gg b_z$  we get

$$\ddot{b}_\perp = \frac{\omega_\perp^2}{b_\perp^{1+2\gamma}}, \quad \ddot{b}_z = \frac{\omega_z^2}{b_\perp^{2\gamma}}. \quad (17)$$

The solution of these equations for a very asymmetric trap ( $\lambda = 0.045$ ) is shown in Fig. 2. We observe the usual elliptic flow phenomenon: the transverse pressure exceeds the longitudinal pressure, there is more acceleration in the transverse direction, and as a consequence the transverse expansion is faster than the longitudinal one. The right panel shows that during the expansion internal energy is converted to kinetic energy of the flow. We can also compute the amount of energy dissipated due to viscous effects. We find

$$\dot{E} = -\frac{4}{3} \left( \frac{\dot{b}_\perp}{b_\perp} - \frac{\dot{b}_z}{b_z} \right)^2 \int d^3x \, \eta(x). \quad (18)$$

An estimate of this quantity is also shown in the right panel of Fig. 2. We show two curves, corresponding to  $\eta/s$  equal to one and three times the conjectured viscosity bound, respectively. We have taken the entropy  $s$  at  $T = T_c$ . We observe that most



**Fig. 2.** The left panel shows the evolution of the scale factors  $b_{\perp}$  and  $b_z$  as a function of the dimensionless variable  $\omega_{\perp} t$ . The right panel shows the time evolution of the internal and kinetic energies. We also show an estimated of the energy dissipated by viscous effects. The two curves correspond to  $\eta/s$  equal to one and three times the conjectured viscosity bound (using  $s$  at  $T = T_c$ ).

of the energy is dissipated early during the expansion. The total energy dissipated amounts to a few percent of the total energy available. This should make the effect observable, although it seems unlikely that a measurement of  $\eta/s$  based on elliptic flow can be as accurate as the one based on collective modes. A measurement of elliptic flow was reported in [27], but there is no systematic study of the temperature dependence of the effect.

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## References

1. I. Arsene et al. [Brahms], B. Back et al. [Phobos], K. Adcox et al. [Phenix], J. Adams et al. [Star], “First Three Years of Operation of RHIC”, Nucl. Phys. **A757**, 1-183 (2005).
2. J. P. Blaizot, E. Iancu and A. Rebhan, Thermodynamics of the high temperature quark gluon plasma, in Quark Gluon Plasma 3, R. Hwa, X.-N. Wang, eds., (2003) [hep-ph/0303185].
3. P. Arnold, G. D. Moore and L. G. Yaffe, JHEP **0305**, 051 (2003) [hep-ph/0302165].
4. D. Teaney, Phys. Rev. C **68**, 034913 (2003) [nucl-th/0301099].
5. T. Hirano and M. Gyulassy, Nucl. Phys. A **769**, 71 (2006) [nucl-th/0506049].
6. P. Danielewicz and M. Gyulassy, Phys. Rev. D **31**, 53 (1985).
7. J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998) [hep-th/9711200].
8. S. S. Gubser, I. R. Klebanov and A. A. Tseytlin, Nucl. Phys. B **534**, 202 (1998) [hep-th/9805156].
9. G. Policastro, D. T. Son and A. O. Starinets, Phys. Rev. Lett. **87**, 081601 (2001) [hep-th/0104066].

10. P. Kovtun, D. T. Son and A. O. Starinets, Phys. Rev. Lett. **94**, 111601 (2005) [hep-th/0405231].
11. There are some puzzles related to the viscosity bound in non-relativistic systems, see T. D. Cohen, hep-th/0702136; A. Dobado and F. J. Llanes Estrada, hep-th/0703132.
12. C. Regal, Ph. D. Thesis, University of Colorado (2005), cond-mat/0601054.
13. P. Massignan, G. M. Bruun, H. Smith, Phys. Rev. A **71**, 033607 (2005) [cond-mat/0409660].
14. G. M. Kavoulakis, C. J. Pethick, H. Smith, Phys. Rev. A **57**, 2938 (1998) [cond-mat/9710130];
15. B. A. Gelman, E. V. Shuryak, and I. Zahed, Phys. Rev. A **72**, 043601 (2005) [nucl-th/0410067].
16. T. Schäfer, preprint, cond-mat/0701251.
17. J. Carlson, S. Y. Chang, V. R. Pandharipande, K. E. Schmidt, Phys. Rev. Lett. **91**, 50401 (2003).
18. H. Heiselberg, Phys. Rev. Lett. **93**, 040402 (2004) [cond-mat/0403041];
19. S. Stringari, Europhys. Lett. **65**, 749 (2004) [cond-mat/0312614].
20. A. Bulgac and G. F. Bertsch, Phys. Rev. Lett. **94**, 070401 (2005) [cond-mat/0404687].
21. J. Kinast, S. L. Hemmer, M. E. Gehm, A. Turlapov, and J. E. Thomas, Phys. Rev. Lett. **92**, 150402 (2004); J. Kinast, A. Turlapov, J. E. Thomas, Phys. Rev. A **70**, 051401(R) (2004); M. Bartenstein, A. Altmeyer, S. Riedl, S. Jochim, C. Chin, J. Hecker Denschlag, and R. Grimm, Phys. Rev. Lett. **92**, 203201 (2004) [cond-mat/0412712]; A. Altmeyer, S. Riedl, C. Kohstall, M. Wright, R. Geursen, M. Bartenstein, C. Chin, J. Hecker Denschlag, R. Grimm, preprint, cond-mat/0609390.
22. J. E. Thomas, J. Kinast, A. Turlapov, Phys. Rev. Lett. **95**, 120402 (2005) [cond-mat/0503620].
23. J. Kinast, A. Turlapov, J. E. Thomas, Phys. Rev. Lett. **94**, 170404 (2005) [cond-mat/0502507].
24. J. Kinast, A. Turlapov, J. E. Thomas, Q. Chen, J. Stajic, and K. Levin, Science **307**, 1296 (2005) [cond-mat/0502087].
25. L. Luo, B. Clancy, J. Joseph, J. Kinast, J. E. Thomas, preprint, cond-mat/0611566.
26. C. Menotti, P. Pedri, S. Stringari, Phys. Rev. Lett. **89**, 250402 (2002) [cond-mat/0208150].
27. K. M. O'Hara, S. L. Hemmer, M. E. Gehm, S. R. Granade, J. E. Thomas, Science **298**, 2179 (2002) [cond-mat/0212463].